

# THE SECOND AND THIRD MOMENT OF $L(\frac{1}{2}, \chi)$ IN THE HYPERELLIPTIC ENSEMBLE

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**ABSTRACT.** We obtain asymptotic formulas for the second and third moment of quadratic Dirichlet  $L$ -functions at the critical point, in the function field setting. We fix the ground field  $\mathbb{F}_q$ , and assume for simplicity that  $q$  is a prime with  $q \equiv 1 \pmod{4}$ . We compute the second and third moment of  $L(1/2, \chi_D)$  when  $D$  is a monic, square-free polynomial of degree  $2g + 1$ , as  $g \rightarrow \infty$ . The answer we get for the second moment agrees with Andrade and Keating's conjectured formula in [3]. For the third moment, we check that the leading term agrees with the conjecture.

## 1. INTRODUCTION

In this paper, we study the second and third moment of quadratic Dirichlet  $L$ -functions in the function field setting. We obtain asymptotic formulas for

$$\sum_{D \in \mathcal{H}_{2g+1}} L\left(\frac{1}{2}, \chi_D\right)^k,$$

when  $k = 2, 3$  and  $g \rightarrow \infty$ , where  $\mathcal{H}_{2g+1}$  denotes the space of monic, square-free polynomials of degree  $2g + 1$  over  $\mathbb{F}_q[x]$ . In our calculation, we take  $q$  to be a prime with  $q \equiv 1 \pmod{4}$ . More precisely, we prove the following.

**Theorem 1.1.** *Let  $q$  be a prime with  $q \equiv 1 \pmod{4}$ . Then*

$$\sum_{D \in \mathcal{H}_{2g+1}} L\left(\frac{1}{2}, \chi_D\right)^2 = \frac{q^{2g+1}}{\zeta(2)} P(2g+1) + O(q^{g(1+\epsilon)}),$$

where  $\mathcal{H}_{2g+1}$  denotes the space of monic, square-free polynomials of degree  $2g + 1$  over  $\mathbb{F}_q[x]$ ,  $\zeta$  is the zeta function associated with  $\mathbb{F}_q[x]$  and  $P(x)$  is a polynomial of degree 3 whose coefficients will be computed explicitly.

**Theorem 1.2.** *Under the same assumptions as above, we have that*

$$\sum_{D \in \mathcal{H}_{2g+1}} L\left(\frac{1}{2}, \chi_D\right)^3 = \frac{q^{2g+1}}{\zeta(2)} Q(2g+1) + O(q^{3g/2(1+\epsilon)}),$$

where  $Q(x)$  is a polynomial of degree 6 whose coefficients can be computed explicitly.

There has been a long-standing interest in understanding moments of families of  $L$ -functions. For the zeta-function, if we define

$$M_k(T) = \frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt,$$

the problem is to find asymptotic formulas for  $M_k$  as  $T \rightarrow \infty$ . The leading term for the second moment was computed in [8] to be

$$M_1 \sim \log T,$$

and the fourth moment leading term was computed by Ingham [10]

$$M_2(T) \sim \frac{1}{2\pi^2} \log^4(T).$$

No other higher moments have been computed so far, but it is conjectured that

$$M_k(T) \sim C_k (\log T)^{k^2},$$

for all  $k > 0$ . A precise value for  $C_k$  was conjectured by Keating and Snaith [13] using random matrix theory.

One can look at other families of  $L$ -functions. For example, considering the family of Dirichlet  $L$ -functions  $L(s, \chi_d)$ , we are interested in

$$(1.1) \quad \sum_{0 < d \leq D} L\left(\frac{1}{2}, \chi_d\right)^k,$$

where the sum is over real primitive Dirichlet characters. It is conjectured that the  $k^{\text{th}}$  moment above is asymptotic to  $C_k D (\log D)^{k(k+1)/2}$ . Jutila [11] computed the first and second moment and Soundararajan [16] computed the second moment  $\sum L(\frac{1}{2}, \chi_{8d})^2$  and the third moment  $\sum L(\frac{1}{2}, \chi_{8d})^3$ , where the sum is over square-free, odd, positive  $d$ . Keating and Snaith [12] conjectured the leading term for (1.1), again using random matrix theory. The other principal lower order terms have been conjectured by Conrey, Farmer, Keating, Rubinstein and Snaith in [5].

In [6], Diaconu, Goldfeld and Hoffstein use multiple Dirichlet series to study the moments of  $L(1/2, \chi_d)$ . Their work suggests the existence of a lower order term of size  $X^{3/4}$  for the cubic moment. Zhang [19] conjectured a value for the constant associated with this term. Young [18] considered the smoothed third moment of this family of  $L$ -functions and bounded the remainder term by  $O(X^{3/4+\epsilon})$ .

In this paper, we are interested in the analogous problem of moments of  $L$ -functions over function fields. Andrade and Keating [2] computed the mean value of  $L(1/2, \chi_D)$  averaged over monic square-free polynomials of degree  $2g + 1$ . When the cardinality of the field  $\mathbb{F}_q$  is  $q \equiv 1 \pmod{4}$ , they proved that

$$(1.2) \quad \sum_{D \in \mathcal{H}_{2g+1}} L(\tfrac{1}{2}, \chi_D) = \frac{P(1)}{2\zeta(2)} q^{2g+1} \left[ (2g+1) + 1 + \frac{4}{\log q} \frac{P'}{P}(1) \right] + O(q^{(2g+1)(3/4 + \frac{\log_q 2}{2})}),$$

where  $\mathcal{H}_{2g+1}$  denotes the space of monic, square-free polynomials of degree  $2g + 1$  over  $\mathbb{F}_q[x]$  and

$$P(s) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{(|P|+1)|P|^s} \right).$$

Extending the recipe in [5] to the function field setting, Andrade and Keating [3] conjectured formulas for integral moments of  $L$ -functions over function fields. More precisely, they conjectured that

$$(1.3) \quad \sum_{D \in \mathcal{H}_{2g+1}} L(\tfrac{1}{2}, \chi_D)^k = \sum_{D \in \mathcal{H}_{2g+1}} Q_k(2g+1)(1 + o(1)),$$

where  $Q_k$  is a polynomial of degree  $k(k+1)/2$  given by

$$Q_k(x) = \frac{(-1)^{k(k-1)/2} 2^k}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{G(z_1, \dots, z_k) \Delta(z_1^2, \dots, z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} q^{\frac{x}{2} \sum_{j=1}^k z_j} dz_1 \cdots dz_k,$$

and

$$G(z_1, \dots, z_k) = A\left(\frac{1}{2}; z_1, \dots, z_k\right) \prod_{j=1}^k X\left(\frac{1}{2} + z_j\right)^{-1/2} \prod_{1 \leq i \leq j \leq k} \zeta(1 + z_i + z_j).$$

In the above,

$$X(s) = q^{-1/2+s},$$

and

$$(1.4) \quad A\left(\frac{1}{2}; z_1, \dots, z_k\right) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \prod_{1 \leq i \leq j \leq k} \left(1 - \frac{1}{|P|^{1+z_i+z_j}}\right) \\ \times \left(\frac{1}{2} \left(\prod_{j=1}^k \left(1 - \frac{1}{|P|^{1/2+z_j}}\right)^{-1} + \prod_{j=1}^k \left(1 + \frac{1}{|P|^{1/2+z_j}}\right)^{-1}\right) + \frac{1}{|P|}\right) \left(1 + \frac{1}{|P|}\right)^{-1}.$$

For  $k = 1$ , the conjecture above agrees with the computed first moment (1.2). Recently, Rubinstein and Wu [15] provided numerical evidence in favor of these conjectures.

When  $k = 2$ , the conjectured formula (1.3) simplifies to  $\sum_{D \in \mathcal{H}_{2g+1}} L(\frac{1}{2}, \chi_D)^2 = \frac{q^{2g+1}}{\zeta(2)} R(2g + 1) + o(q^{2g+1})$ , where

$$(1.5) \quad R(x) = \frac{1}{24 \log(q)^3} \left[ (6 + 11x + 6x^2 + x^3) A(0, 0) (\log q)^3 \right. \\ \left. + (11 + 12x + 3x^2) (\log q)^2 (A_2(0, 0) + A_1(0, 0)) + 12(2 + x) (\log q) A_{12}(0, 0) \right. \\ \left. - 2(A_{222}(0, 0) - 3A_{122}(0, 0) - 3A_{112}(0, 0) + A_{111}(0, 0)) \right],$$

where the  $A_j$  above are partial derivatives of  $A(1/2; z_1, z_2)$  evaluated at  $z_1 = z_2 = 0$ . Our answer in Theorem 1.1 agrees with the conjecture (1.5).

For the third moment, Andrade and Keating [3] conjecture that

$$(1.6) \quad \sum_{D \in \mathcal{H}_{2g+1}} L(\frac{1}{2}, \chi_D)^3 \sim \frac{1}{2880 \zeta(2)} A_3(\frac{1}{2}; 0, 0, 0) |D| (\log_q |D|)^6,$$

with

$$(1.7) \quad A_3(\frac{1}{2}; 0, 0, 0) = \prod_P \left(1 - \frac{12|P|^5 - 23|P|^4 + 23|P|^3 - 15|P|^2 + 6|P| - 1}{|P|^6(|P| + 1)}\right).$$

We obtain an asymptotic formula for the third moment with an error of size  $O(q^{3g/2(1+\epsilon)})$  and we check that the leading term agrees with (1.6). Checking by hand that all the other lower order terms match the conjecture (1.3) involves laborious computations, and we do not carry them out here.

## 2. BACKGROUND AND SETUP OF THE PROBLEM

We introduce the notation we use throughout the paper. Let  $\mathcal{M}$  denote the monic polynomials over  $\mathbb{F}_q[x]$ ,  $\mathcal{M}_n$  the monic polynomials of degree  $n$  over  $\mathbb{F}_q[x]$  and  $\mathcal{M}_{\leq n}$  the monic polynomials of degree less than or equal to  $n$ . Then  $|\mathcal{M}_n| = q^n$  and  $|\mathcal{M}_{\leq n}| = 1 + q + \dots + q^n = (q^{n+1} - 1)/(q - 1)$ .

Let  $\mathcal{H}_{d,q}$  denote the set of monic square-free polynomials of degree  $d$  over  $\mathbb{F}_q[x]$ . For ease of notation, we will write it as  $\mathcal{H}_d$ . The norm of a polynomial  $f \in \mathbb{F}_q[x]$  is defined as  $|f| = q^{d(f)}$ , where

for simplicity,  $d(f) = \deg(f)$ .  $d_k(f)$  will denote the  $k^{\text{th}}$  divisor function (i.e.  $d_k(f) = \sum_{f_1 \cdots f_k = f} 1$ .)

From now on,  $P$  will be used to denote a monic irreducible polynomial.

**2.1. Basic facts about  $L$ -functions over function fields.** Many of the facts stated in this section are proven in [14].

For  $\text{Re}(s) > 1$ , the zeta function of  $\mathbb{F}_q[x]$  is defined by

$$\zeta(s) = \sum_{f \in \mathcal{M}} \frac{1}{|f|^s} = \prod_P (1 - |P|^{-s})^{-1}.$$

One can show that  $\zeta(s) = (1 - q^{1-s})^{-1}$ . With the change of variables  $u = q^{-s}$ , we have  $\mathcal{Z}(u) = (1 - qu)^{-1}$ .

To determine the cardinality of  $\mathcal{H}_n$ , consider the generating series

$$\sum_{\substack{D \text{ monic} \\ \text{square-free}}} u^{d(D)} = \frac{\mathcal{Z}(u)}{\mathcal{Z}(u^2)} = \frac{1 - qu^2}{1 - qu}.$$

Looking at the coefficient of  $u^n$ , we see that for  $n = 1$ ,  $|\mathcal{H}_1| = q$  and for  $n \geq 2$ ,  $|\mathcal{H}_n| = q^n(1 - 1/q) = q^n/\zeta(2)$ .

For a monic irreducible polynomial  $P$ , define the quadratic residue  $\left(\frac{f}{P}\right)$  by

$$\left(\frac{f}{P}\right) = \begin{cases} 1 & \text{if } P \nmid f \text{ and } f \text{ is a square (mod } P) \\ -1 & \text{if } P \nmid f \text{ and } f \text{ is not a square (mod } P) \\ 0 & \text{if } P \mid f. \end{cases}$$

If  $Q = P_1^{e_1} \cdots P_k^{e_k}$  is the prime factorization of  $Q$  in  $\mathbb{F}_q[x]$ , then the Jacobi symbol is defined by

$$\left(\frac{f}{Q}\right) = \prod_{i=1}^k \left(\frac{f}{P_i}\right)^{e_i}.$$

Artin proved the quadratic reciprocity law over function fields, namely that if  $A, B \in \mathbb{F}_q[x]$  are non-zero, relatively prime monic polynomials, then

$$\left(\frac{A}{B}\right) = \left(\frac{B}{A}\right) (-1)^{((q-1)/2)d(A)d(B)}.$$

For  $D \in \mathbb{F}_q[x]$ , the Dirichlet character  $\chi_D$  is defined by

$$\chi_D(f) = \left(\frac{D}{f}\right).$$

The  $L$ -function associated to  $\chi_D$  is defined by

$$L(s, \chi_D) = \sum_{f \in \mathcal{M}} \frac{\chi_D(f)}{|f|^s} = \prod_P (1 - \chi_D(P)|P|^{-s})^{-1}.$$

This converges for  $\text{Re}(s) > 1$ . Using the change of variables  $u = q^{-s}$ ,

$$\mathcal{L}(u, \chi_D) = \prod_P (1 - \chi_D(P)u^{d(P)})^{-1}.$$

One can show that when  $D$  is a non-square polynomial,  $\mathcal{L}(u, \chi_D)$  is a polynomial in  $u$  of degree at most  $d(D) - 1$ .

When  $D$  is a monic square-free polynomial, the completed  $L$ -function is defined by

$$\mathcal{L}(u, \chi_D) = (1 - u)^\lambda \mathcal{L}^*(u, \chi_D),$$

where

$$\lambda = \begin{cases} 1 & \text{if } d(D) \text{ even} \\ 0 & \text{if } d(D) \text{ odd} \end{cases}$$

Then  $\mathcal{L}^*(u, \chi_D)$  is a polynomial of degree  $2\delta = d(D) - 1 - \lambda$  and satisfies the functional equation

$$\mathcal{L}^*(u, \chi_D) = (qu^2)^\delta \mathcal{L}^*(1/(qu), \chi_D).$$

In particular, if  $D \in \mathcal{H}_{2g+1}$ , then  $\mathcal{L}(u, \chi_D)$  is a polynomial of degree  $2g$  satisfying the above functional equation.

We can relate the  $L$ -function to zeta functions of curves. If  $C$  is a smooth, projective, geometrically connected curve of genus  $g$  over  $\mathbb{F}_q$ , then the zeta function of  $C$  is defined by

$$Z_C(u) = \exp \left( \sum_{r=1}^{\infty} N_r(C) \frac{u^r}{r} \right),$$

where  $N_r(C)$  is the number of points on  $C$  with coordinates in  $\mathbb{F}_{q^r}$ . Weil [17] proved that the zeta function of  $C$  is a rational function, equal to

$$Z_C(u) = \frac{P_C(u)}{(1-u)(1-qu)},$$

where  $P_C(u)$  is a polynomial of degree  $2g$ . The Riemann hypothesis for curves over finite fields was proven by Weil [17] and states that the zeros of the polynomial  $P_C(u)$  all lie on the circle  $|u| = q^{-1/2}$ .

When  $D$  is monic and square-free, the equation  $y^2 = D(x)$  defines a projective, connected, hyperelliptic curve. The polynomial  $P_{C_D}(u)$  that appears in the zeta function of  $C_D$  coincides with the completed  $L$ -function  $\mathcal{L}^*(u, \chi_D)$ , as proven in Artin's thesis.

**2.2. Preliminary lemmas.** We will quote a number of lemmas we will use in the paper. We assume for simplicity that  $q$  is a prime with  $q \equiv 1 \pmod{4}$ .

The following exact formula is an analogue of the approximate functional equation for  $L(1/2, \chi_d)$  in the number field setting.

**Lemma 2.1.** *Let  $D \in \mathcal{H}_{2g+1}$ . For  $k$  an integer, we have the following functional equation:*

$$L(\tfrac{1}{2}, \chi_D)^k = \sum_{f \in \mathcal{M}_{\leq kg}} \frac{\chi_D(f) d_k(f)}{\sqrt{|f|}} + \sum_{f \in \mathcal{M}_{\leq kg-1}} \frac{\chi_D(f) d_k(f)}{\sqrt{|f|}},$$

where  $d_k$  is the  $k^{\text{th}}$  divisor function.

*Proof.* The proof is similar to the proof of the functional equation of  $L(1/2, \chi_P)^2$ , with  $P$  a monic irreducible polynomial in [4] and we will omit it.  $\square$

We also need the following lemma, whose proof can be found in [7].

**Lemma 2.2.** *For  $f$  a monic polynomial in  $\mathbb{F}_q[x]$ , we have that*

$$\sum_{D \in \mathcal{H}_{2g+1}} \chi_f(D) = \sum_{C|f^\infty} \sum_{h \in \mathcal{M}_{2g+1-2d(C)}} \chi_f(h) - q \sum_{C|f^\infty} \sum_{h \in \mathcal{M}_{2g-1-2d(C)}} \chi_f(h),$$

where the first sum is over monic polynomials  $C$  whose prime factors are among the prime factors of  $f$ .

We will now state a version of Poisson summation over function fields. Recall the exponential function introduced in [9]. For  $a \in \mathbb{F}_q((1/x))$ , let

$$e(a) = e^{2\pi i a_1/q},$$

where  $a_1$  is the coefficient of  $1/x$  in the expansion of  $a$  (for more details, see [9].) For  $\chi$  a general character  $(\bmod f)$ , define the generalized Gauss sum as

$$G(V, \chi) = \sum_{u \pmod{f}} \chi(u) e\left(\frac{uV}{f}\right).$$

The following Poisson summation formula holds.

**Lemma 2.3.** *Let  $f$  be a monic polynomial of degree  $n$  in  $\mathbb{F}_q[x]$  and  $m$  a positive integer. If  $d(f)$  is even, then*

$$\sum_{g \in \mathcal{M}_m} \chi_f(g) = \frac{q^m}{|f|} \left[ G(0, \chi_f) + (q-1) \sum_{V \in \mathcal{M}_{\leq n-m-2}} G(V, \chi_f) - \sum_{V \in \mathcal{M}_{n-m-1}} G(V, \chi_f) \right].$$

If  $d(f)$  is odd, then

$$\sum_{g \in \mathcal{M}_m} \chi_f(g) = \frac{q^m}{|f|} \sqrt{q} \sum_{V \in \mathcal{M}_{n-m-1}} G(V, \chi_f).$$

*Proof.* See Proposition 3.1 in [7]. □

We also need to compute the generalized Gauss sums. The proof is similar to the proof of Lemma 2.3 in [16] and we will skip it.

**Lemma 2.4.** *Let  $q$  be a prime with  $q \equiv 1 \pmod{4}$ .*

- (1) *If  $(f, g) = 1$ , then  $G(V, \chi_{fg}) = G(V, \chi_f)G(V, \chi_g)$ .*
- (2) *Suppose  $V = V_1 P^\alpha$  where  $P \nmid V_1$ . Then*

$$G(V, \chi_{P^i}) = \begin{cases} 0 & \text{if } i \leq \alpha \text{ and } i \text{ odd} \\ \phi(P^i) & \text{if } i \leq \alpha \text{ and } i \text{ even} \\ -|P|^{i-1} & \text{if } i = \alpha + 1 \text{ and } i \text{ even} \\ \left(\frac{V_1}{P}\right) |P|^{i-1} |P|^{1/2} & \text{if } i = \alpha + 1 \text{ and } i \text{ odd} \\ 0 & \text{if } i \geq 2 + \alpha. \end{cases}$$

**2.3. Outline of the proof.** We will use the functional equation for  $L(1/2, \chi_D)^k$  (with  $k = 2, 3$ ) as given in Lemma 2.1, and then Lemma 2.2 to transform the sum over square-free polynomials  $D$  into sums involving monic polynomials. We'll use the Poisson summation formula as in Lemma 2.3 for these sums, getting another summation over monic polynomials  $V$ .

We will first focus on the second moment of  $L(1/2, \chi_D)$ . There will be a main term of size  $q^{2g+1}(2g+1)^3$  coming from the contribution of square polynomials  $f$  in the functional equation, which we will evaluate in section 3. Unlike the case of the mean value of  $L(1/2, \chi_D)$  in the hyperelliptic ensemble, there will be another secondary main term, which will come from the contribution of square polynomials  $V$  (where  $V$  is the dual variable in the Poisson summation formula). The secondary main term is of size  $q^{2g+1}(2g+1)$ , and we will explicitly compute it in section 4. We note that computing the secondary main term is the most delicate part of the proof, and it reduces to exactly evaluating a certain contour integral, which can be done by using a functional equation of the integrand.

We will then evaluate the sum over non-square polynomials  $V$  in section 5 and show that it is bounded by  $q^{g(1+\epsilon)}$ . In section 6 we put together the main term and the secondary main term and check that our answer agrees with the conjecture (1.5).

We use the same methods to evaluate the third moment in section 7. Since most of the computations are very similar to the ones carried out before, we will only briefly sketch the proof. The main term corresponding to square polynomials  $f$  is of size  $q^{2g+1}(2g+1)^6$ , and the secondary main term, coming from square  $V$ , is also of size  $q^{2g+1}(2g+1)^6$ . We note that evaluating the secondary main term for the third moment again reduces to computing a certain contour integral, which is easier to do than in the second moment case. Here, by simply shifting contours, we get a main term and an error of size  $q^{3g/2(1+\epsilon)}$ . Bounding the contribution from non-square polynomials  $V$  is similar to the method used for the second moment.

**2.4. Setup of the problem.** In what follows,  $k = 2, 3$ . Using the functional equation in Lemma 2.1 and Lemma 2.2 it follows that

$$\sum_{D \in \mathcal{H}_{2g+1}} L\left(\frac{1}{2}, \chi_D\right)^k = S_{kg} + S_{kg-1},$$

where

$$S_{kg} = \sum_{f \in \mathcal{M}_{\leq kg}} \frac{d_k(f)}{\sqrt{|f|}} \sum_{\substack{C \in \mathcal{M}_{\leq g} \\ C|f^\infty}} \sum_{h \in \mathcal{M}_{2g+1-2d(C)}} \chi_f(h) - q \sum_{f \in \mathcal{M}_{\leq kg}} \frac{d_k(f)}{\sqrt{|f|}} \sum_{\substack{C \in \mathcal{M}_{\leq g-1} \\ C|f^\infty}} \sum_{h \in \mathcal{M}_{2g-1-2d(C)}} \chi_f(h).$$

Similarly as in [7], the term in the expression for  $S_{kg}$  corresponding to  $C \in \mathcal{M}_g$  is bounded by  $O(q^{kg/2(1+\epsilon)})$ . Then we rewrite

$$S_{kg} = \sum_{f \in \mathcal{M}_{\leq kg}} \frac{d_k(f)}{\sqrt{|f|}} \sum_{\substack{C \in \mathcal{M}_{\leq g-1} \\ C|f^\infty}} \left( \sum_{h \in \mathcal{M}_{2g+1-2d(C)}} \chi_f(h) - q \sum_{h \in \mathcal{M}_{2g-1-2d(C)}} \chi_f(h) \right) + O(q^{kg/2(1+\epsilon)}).$$

A similar expression holds for  $S_{kg-1}$ . We'll focus on  $S_{kg}$ . Write  $S_{kg} = S_{kg,e} + S_{kg,o} + O(q^{kg/2(1+\epsilon)})$ , where  $S_{kg,e}$  is the sum over polynomials  $f$  of even degree less than or equal to  $kg$ , and  $S_{kg,o}$  the sum over polynomials  $f$  of odd degree. When summing over polynomials of even degree, we use the Poisson summation formula in Lemma 2.3 for the sum over  $h$ , and let  $M_{kg}$  be the term corresponding to  $V = 0$ . Note that using Lemma 2.4,  $G(0, \chi_f)$  is nonzero if and only if  $f$  is a square, in which case  $G(0, \chi_f) = \phi(f)$ . Write  $S_{kg,e} = M_{kg} + S_{kg,e}(V \neq 0)$ , where

$$M_{kg} = q^{2g+1} \left(1 - \frac{1}{q}\right) \sum_{\substack{f \in \mathcal{M}_{\leq kg} \\ f = \square}} \frac{d_k(f)}{|f|^{\frac{3}{2}}} \phi(f) \sum_{\substack{C \in \mathcal{M}_{\leq g-1} \\ C|f^\infty}} \frac{1}{|C|^2},$$

$$\begin{aligned} S_{kg,e}(V \neq 0) &= q^{2g+1} \sum_{\substack{f \in \mathcal{M}_{\leq kg} \\ d(f) \text{ even}}} \frac{d_k(f)}{|f|^{\frac{3}{2}}} \sum_{\substack{C \in \mathcal{M}_{\leq g-1} \\ C|f^\infty}} \frac{1}{|C|^2} \left[ (q-1) \sum_{V \in \mathcal{M}_{\leq d(f)-2g-3+2d(C)}} G(V, \chi_f) - \sum_{V \in \mathcal{M}_{d(f)-2g-2+2d(C)}} G(V, \chi_f) \right. \\ (2.1) \quad &\quad \left. - \frac{q-1}{q} \sum_{V \in \mathcal{M}_{\leq d(f)-2g-1+2d(C)}} G(V, \chi_f) + \frac{1}{q} \sum_{V \in \mathcal{M}_{d(f)-2g+2d(C)}} G(V, \chi_f) \right]. \end{aligned}$$

Again using the Poisson summation formula in 2.3,  
(2.2)

$$S_{kg,o} = q^{2g+1} \sqrt{q} \sum_{\substack{f \in \mathcal{M}_{\leq kg} \\ d(f) \text{ odd}}} \frac{d_k(f)}{|f|^{\frac{3}{2}}} \sum_{\substack{C \in \mathcal{M}_{\leq g-1} \\ C|f^\infty}} \frac{1}{|C|^2} \left( \sum_{V \in \mathcal{M}_{d(f)-2g-2+2d(C)}} G(V, \chi_f) - \frac{1}{q} \sum_{V \in \mathcal{M}_{d(f)-2g+2d(C)}} G(V, \chi_f) \right).$$

In equation (2.1), we write the sum over  $V$  as the sum over square  $V$  plus the sum over non-square  $V$ . Let  $S_{kg,e}(V \neq 0) = S_{kg}(V = \square) + S_{kg,e}(V \neq \square)$ .

When  $V$  is a square, write  $V = l^2$ . Using equation (2.1), we rewrite

$$(2.3) \quad S_{kg}(V = \square) = q^{2g+1} \sum_{\substack{f \in \mathcal{M}_{\leq kg} \\ d(f) \text{ even}}} \frac{d_k(f)}{|f|^{\frac{3}{2}}} \sum_{\substack{C \in \mathcal{M}_{\leq g-1} \\ C|f^\infty}} |C|^{-2} \left[ (q-1) \sum_{l \in \mathcal{M}_{\leq \frac{d(f)}{2}-g-2+d(C)}} G(l^2, \chi_f) - \sum_{l \in \mathcal{M}_{\frac{d(f)}{2}-g-1+d(C)}} G(l^2, \chi_f) \right. \\ \left. - \frac{q-1}{q} \sum_{l \in \mathcal{M}_{\leq \frac{d(f)}{2}-g-1+d(C)}} G(l^2, \chi_f) + \frac{1}{q} \sum_{l \in \mathcal{M}_{\frac{d(f)}{2}-g+d(C)}} G(l^2, \chi_f) \right].$$

Let  $S_k(V = \square) = S_{kg}(V = \square) + S_{kg-1}(V = \square)$  (where  $S_{kg-1}(V = \square)$  is defined in the same way as  $S_{kg}(V = \square)$ ). We'll show that  $S_k(V = \square)$  (which is the secondary main term) is of size  $gq^{2g+1}$  when  $k = 2$  and of size  $g^6 q^{2g+1}$  when  $k = 3$ . Define  $S_{kg}(V \neq \square) = S_{kg,o} + S_{kg,e}(V \neq \square)$ , with  $S_{kg,o}$  given by (2.2) and  $S_{kg,e}(V \neq \square)$  the sum over non-square polynomials  $V$  in (2.1). Similarly define  $S_{kg-1}(V \neq \square)$ . We'll bound  $S_{kg}(V \neq \square)$  and  $S_{kg-1}(V \neq \square)$  by  $O(q^{kg/2(1+\epsilon)})$ .

### 3. MAIN TERM

In the next four sections, we will concentrate on the second moment of  $L(1/2, \chi_D)$ . Here, we focus on evaluating the main term corresponding to the contribution of square polynomials  $f$ . Recall that

$$(3.1) \quad M_{2g} = q^{2g+1} \left(1 - \frac{1}{q}\right) \sum_{\substack{f \in \mathcal{M}_{\leq 2g} \\ f = \square}} \frac{d_2(f)}{|f|^{\frac{3}{2}}} \phi(f) \sum_{\substack{C \in \mathcal{M}_{\leq g-1} \\ C|f^\infty}} \frac{1}{|C|^2}.$$

A similar expression holds for  $M_{2g-1}$ . The main term  $M_{2g} + M_{2g-1}$  is given by the following lemma.

**Lemma 3.1.** *Using the same notation as before, we have*

$$M_{2g} + M_{2g-1} = \frac{q^{2g+1}}{\zeta(2)} P_1(2g+1) + O(q^{g(1+\epsilon)}),$$

where  $P_1$  is the polynomial of degree 3 given by (3.7).

To prove this, we express  $M_{2g}$  and  $M_{2g-1}$  as contour integrals and then evaluate them. We do so in the next lemma.

**Lemma 3.2.** *With the same notation as before, we have*

$$M_{2g} = \frac{q^{2g+1}}{\zeta(2)} \frac{1}{2\pi i} \oint_{|u|=r_1} \frac{(1-qu^2)\mathcal{B}(u)}{(1-qu)^4(qu)^g} \frac{du}{u} + O(q^{g\epsilon}),$$

and

$$M_{2g-1} = \frac{q^{2g+1}}{\zeta(2)} \frac{1}{2\pi i} \oint_{|u|=r_1} \frac{(1-qu^2)\mathcal{B}(u)}{(1-qu)^4(qu)^{g-1}} \frac{du}{u} + O(q^{g\epsilon}),$$



where

$$(3.2) \quad \mathcal{B}(u) = \prod_P \left( 1 + \frac{u^{d(P)}(u^{d(P)} - 3)}{(|P| + 1)(1 + u^{d(P)})} \right),$$

and  $r_1 < 1/q$ .

*Remark 1.* Note that  $\mathcal{B}(u)$  converges for  $|u| < 1$ .

*Proof.* In equation (3.1), write  $f = l^2$ , with  $l \in \mathcal{M}_m$ . Note that  $C|f^\infty$  if and only if  $C|l^\infty$ . Similarly as in [7], we have

$$\sum_{\substack{C \in \mathcal{M}_{\leq g-1} \\ C|f^\infty}} |C|^{-2} = \sum_{\substack{C \in \mathcal{M}_{\leq g-1} \\ C|l^\infty}} |C|^{-2} = \prod_{P|l} \left( 1 - \frac{1}{|P|^2} \right)^{-1} + O(q^{-g(2-\epsilon)}).$$

Since  $\phi(l^2)/|l|^2 = \prod_{P|l} (1 - |P|^{-1})$  and  $(1 - 1/q)^{-1} = \zeta(2)$ , we have

$$(3.3) \quad M_{2g} = \frac{q^{2g+1}}{\zeta(2)} \sum_{l \in \mathcal{M}_{\leq g}} \frac{d_2(l^2)}{|l| \prod_{P|l} (1 + |P|^{-1})} + O(q^{g\epsilon}).$$

Let

$$\mathcal{A}(u) = \sum_{l \in \mathcal{M}} \frac{d_2(l^2)}{\prod_{P|l} (1 + |P|^{-1})} u^{d(l)}.$$

By multiplicativity, we can write

$$\mathcal{A}(u) = \frac{\mathcal{Z}(u)^3}{\mathcal{Z}(u^2)} \mathcal{B}(u),$$

with  $\mathcal{B}(u)$  given by (7.3). Note that  $\mathcal{Z}(u) = (1 - qu)^{-1}$  and  $\mathcal{Z}(u^2) = (1 - qu^2)^{-1}$ , so

$$(3.4) \quad \mathcal{A}(u) = \frac{1 - qu^2}{(1 - qu)^3} \mathcal{B}(u).$$

Now we will use the following remark, which is the function field analogue of Perron's formula. If the power series  $\sum_{f \in \mathcal{M}} a(f) u^{d(f)}$  converges absolutely for  $|u| \leq R < 1$ , then

$$(3.5) \quad \sum_{f \in \mathcal{M}_{\leq k}} a(f) = \frac{1}{2\pi i} \oint_{|u|=R} \left( \sum_{f \in \mathcal{M}} a(f) u^{d(f)} \right) \frac{u^{-k}}{1 - u} \frac{du}{u}.$$

Using this in (3.3) gives

$$M_{2g} = \frac{q^{2g+1}}{\zeta(2)} \frac{1}{2\pi i} \oint_{|u|=r_1} \frac{(1 - qu^2)\mathcal{B}(u)}{(1 - qu)^4 (qu)^g} \frac{du}{u} + O(q^{g\epsilon}),$$

where  $r_1 < 1/q$ . We can similarly express  $M_{2g-1}$ , which finishes the proof of Lemma 3.2.  $\square$

*Proof of Lemma 3.1.* In Lemma 3.2, note that the integrand  $((1 - qu^2)\mathcal{B}(u))/(u(1 - qu)^4(qu)^g)$  has a pole of order 4 at  $u = 1/q$ . Since  $\mathcal{B}(u)$  converges absolutely for  $|u| < 1$ , we can write

$$\frac{1}{2\pi i} \oint_{|u|=r_1} \frac{(1 - qu^2)\mathcal{B}(u)}{(1 - qu)^4 (qu)^g} \frac{du}{u} = -\text{Res}(u = 1/q) + \frac{1}{2\pi i} \oint_{|u|=r_2} \frac{(1 - qu^2)\mathcal{B}(u)}{(1 - qu)^4 (qu)^g} \frac{du}{u},$$

where  $r_2 = q^{-\epsilon}$ . We can explicitly compute the residue at  $u = 1/q$ , and we can bound the integral on the right-hand side above by

$$\left| \frac{1}{2\pi i} \oint_{|u|=r_2} \frac{(1-qu^2)\mathcal{B}(u)}{(1-qu)^4 u (qu)^g} du \right| \ll q^{-g(1-\epsilon)}.$$

We can similarly express  $M_{2g-1}$  in terms of the residue of the integrand  $((1-qu^2)\mathcal{B}(u))/(u(1-qu)^4(qu)^{g-1})$  at  $u = 1/q$ . Computing the residues at  $u = 1/q$  gives

$$(3.6) \quad M_{2g} + M_{2g-1} = \frac{q^{2g+1}}{\zeta(2)} P_1(2g+1) + O(q^{g(1+\epsilon)}),$$

where  $P_1(x)$  is a polynomial of degree 3. We compute it explicitly as

$$(3.7) \quad \begin{aligned} P_1(x) = & x^3 \frac{\mathcal{B}(1/q)(1-q^{-1})}{24} + x^2 \left[ \frac{\mathcal{B}(1/q)(1+q^{-1})}{4} - \frac{\mathcal{B}'(1/q)(1-q^{-1})}{4q} \right] \\ & + x \left[ \frac{11\mathcal{B}(1/q)(1-q^{-1})}{24} + \frac{3\mathcal{B}'(1/q)(1-q^{-1})}{2q} - \frac{2\mathcal{B}'(1/q)}{q} + \frac{\mathcal{B}''(1/q)(1-q^{-1})}{2q^2} \right] \\ & + \frac{\mathcal{B}(1/q)(1+q^{-1})}{4} - \frac{\mathcal{B}'(1/q)(1-q^{-1})}{4q} + \frac{2\mathcal{B}'(1/q)}{q^2} + \frac{2\mathcal{B}''(1/q)}{q^3} - \frac{\mathcal{B}^{(3)}(1/q)(1-q^{-1})}{3q^3}. \end{aligned}$$

□

#### 4. SECONDARY MAIN TERM

In this section, we will evaluate the secondary main term  $S_2(V = \square)$  coming from the contribution of square polynomials  $V$ . Recall from subsection 2.4 that  $S_2(V = \square) = S_{2g}(V = \square) + S_{2g-1}(V = \square)$ , where  $S_{2g}(V = \square)$  is given by equation (2.3). We will prove the following.

**Lemma 4.1.** *Using the same notation as before, we have that*

$$S_2(V = \square) = \frac{q^{2g+1}}{\zeta(2)} P_2(2g+1) + O(q^{g(1+\epsilon)}),$$

where  $P_2(x)$  is a linear polynomial which can be computed explicitly (see formula (4.1).)

**4.1. A few lemmas.** To prove Lemma 4.1, we will first prove the following results.

**Lemma 4.2.** *Let  $V$  be a monic polynomial in  $\mathbb{F}_q[x]$ . For  $|z| > 1/q^2$ , let*

$$\mathcal{M}(V; z, w) = \sum_{f \in \mathcal{M}} \frac{d_2(f) G(V, \chi_f)}{\sqrt{|f|} \prod_{P|f} \left( 1 - \frac{1}{|P|^2 z^{d(P)}} \right)} w^{d(f)}.$$

(a) *We have*

$$\mathcal{M}(V; z, w) = \mathcal{L}(w, \chi_V)^2 \prod_P \mathcal{M}_P(V; z, w),$$

where

$$\mathcal{M}_P(V; w, u) = \begin{cases} 1 + \frac{2(\frac{V}{P})w^{d(P)}}{|P|^2 z^{d(P)} - 1} + w^{2d(P)} - \frac{4w^{2d(P)}}{1 - \frac{1}{|P|^2 z^{d(P)}}} + \frac{2(\frac{V}{P})w^{3d(P)}}{1 - \frac{1}{|P|^2 z^{d(P)}}} & \text{if } P \nmid V \\ 1 + \left( 1 - \frac{1}{|P|^2 z^{d(P)}} \right)^{-1} \sum_{b=1}^{\infty} \frac{d_2(P^b) G(V, \chi_{P^b})}{|P|^{b/2}} w^{bd(P)} & \text{if } P|V \end{cases}$$

(b) If  $V = l^2$ , with  $l \in \mathcal{M}$ , then

$$\mathcal{M}(l^2; z, w) = Z(w)^2 \prod_P \mathcal{R}_P(l^2; z, w),$$

where

$$\mathcal{R}_P(l^2; z, w) = \begin{cases} 1 + \frac{2w^{d(P)}}{|P|^2 z^{d(P)} - 1} + w^{2d(P)} - \frac{4w^{2d(P)}}{1 - \frac{1}{|P|^2 z^{d(P)}}} + \frac{2w^{3d(P)}}{1 - \frac{1}{|P|^2 z^{d(P)}}} & \text{if } P \nmid l \\ (1 - w^{d(P)})^2 \left( 1 + \left( 1 - \frac{1}{|P|^2 z^{d(P)}} \right)^{-1} \sum_{b=1}^{\infty} \frac{d_2(P^b) G(l^2, \chi_{P^b})}{|P|^{b/2}} w^{bd(P)} \right) & \text{if } P \mid l \end{cases}$$

Moreover,  $\prod_P \mathcal{R}_P(l^2; z, w)$  converges absolutely for  $|w| < q|z|$  and  $|w| < q^{-1/2}$ .

*Proof.* We use the fact that  $G(V, \chi_f)$  is multiplicative as a function of  $f$  and then we manipulate Euler products.  $\square$

**Lemma 4.3.** *Let*

$$\mathcal{R}(z, w) = \sum_{l \in \mathcal{M}} z^{d(l)} \prod_P \mathcal{R}_P(l^2; z, w),$$

with  $\mathcal{R}_P(l^2; z, w)$  defined in Lemma 4.2. Then

$$\mathcal{R}(z, w) = \mathcal{Z}(z) \mathcal{Z}(qw^2 z) \mathcal{Z}\left(\frac{1}{q^2 z}\right) \mathcal{F}(z, w),$$

where  $\mathcal{F}(z, w) = \prod_P \mathcal{F}_P(z, w)$ , with

$$\mathcal{F}_P(z, w) = (1 - w^d)^2 \left( 1 - \frac{1 - 2|P|^2(wz)^d - 2|P|(w^2 z)^d + 2|P|^2(wz^2)^d + (-2|P|^3 + 3|P|^2)(w^2 z^2)^d + |P|^2(w^4 z^2)^d - |P|^3(w^4 z^3)^d}{|P|^2 z^d (1 - |P| w^{2d} z^d)} \right)$$

(here  $d$  stands for  $d(P)$ .)

Moreover,  $\mathcal{F}(z, w)$  is absolutely convergent for  $|z| > 1/q$ ,  $|w| < 1/\sqrt{q}$ ,  $|wz| < 1/q$  and  $|w^2 z| < 1/q^2$ .

*Proof.* We use Lemma 4.2 and then manipulate Euler products.  $\square$

**Lemma 4.4.** *Let  $\alpha(z) = \frac{\frac{1}{q} \frac{d}{dw} \mathcal{F}(z, w)|_{w=1/q}}{\mathcal{F}(z, 1/q)}$ , with  $\mathcal{F}(z, w)$  defined in the previous lemma, and let*

$$\mathcal{F}(z) = \mathcal{F}\left(z, \frac{1}{q}\right).$$

(a) We have

$$\mathcal{F}(z) = \prod_P \left( 1 - \frac{1}{|P|} \right)^2 \left( 1 + \frac{2}{|P|} + \frac{1}{|P|^3} - \frac{1}{|P|^2} (z^{d(P)} + z^{-d(P)}) \right),$$

and  $\mathcal{F}(z) = \mathcal{F}(1/z)$ .

(b) We have

$$\alpha(z) = \sum_P \frac{2d(P)(|P|^2 + z^{d(P)}(-3|P|^3 - 3|P| + |P|^2) + z^{2d(P)}(|P|^4 - |P|^3 + 4|P|^2 - |P| + 2) + z^{3d(P)}(|P|^2 - 2|P|))}{(|P| - 1)(z^{d(P)} - |P|)(|P| - z^{d(P)} - 2|P|^2 z^{d(P)} - |P|^3 z^{d(P)} + |P| z^{2d(P)})},$$

and

$$\alpha(1/z) = \alpha(z) - \frac{2(1+z)}{1-z} - 4z \frac{\mathcal{F}'(z)}{\mathcal{F}(z)}.$$

*Proof.* The first part follows directly by computation from Lemma 4.3. For the second part, we rewrite

$$\alpha(z) = \sum_P \frac{2d(P)}{\frac{|P|}{z^{d(P)}} - 1} - g(z) - 4h(z) = \frac{2z}{1-z} - g(z) - 4h(z),$$

with

$$g(z) = \sum_P \frac{d(P)(6z^{d(P)} - 4|P|z^{d(P)} + 6|P|^2z^{d(P)} - 2|P| - 2|P|z^{2d(P)})}{(|P| - 1)(z^{d(P)} + 2|P|^2z^{d(P)} + |P|^3z^{d(P)} - |P| - |P|z^{2d(P)})},$$

and

$$h(z) = \sum_P \frac{d(P)|P|z^{2d(P)}}{z^{d(P)} + 2|P|^2z^{d(P)} + |P|^3z^{d(P)} - |P| - |P|z^{2d(P)}}.$$

Note that from the definition of  $g(z)$  and  $h(z)$  and from the expression for  $\mathcal{F}(z)$ , we have  $g(z) = g(1/z)$  and

$$h(1/z) = h(z) + z \frac{\mathcal{F}'(z)}{\mathcal{F}(z)}.$$

Combining these, the conclusion now follows.  $\square$

**4.2. Proof of Lemma 4.1.** We now begin the proof of Lemma 4.1. Recall the formula (2.3) for  $S_{2g}(V = \square)$ . Using (3.5) twice, we have

$$S_{2g}(V = \square) = q^{2g+1} \frac{1}{2\pi i} \oint_{|z|=r_1} \sum_{\substack{f \in \mathcal{M}_{\leq 2g} \\ d(f) \text{ even}}} \frac{1}{|f|} \sum_{\substack{C \in \mathcal{M}_{\leq g-1} \\ C|f^\infty}} \frac{(qz-1)}{(1-z)z^{\frac{d(f)}{2}-g+d(C)}} \left( \sum_{l \in \mathcal{M}} z^{d(l)} \frac{d_2(f)G(l^2, \chi_f)}{\sqrt{|f|}} \right) \left( 1 - \frac{1}{qz} \right) dz,$$

where we pick  $r_1 = q^{-1-\epsilon}$ . We can extend the sum over  $C|f^\infty$  with  $d(C) \leq g-1$  to include all polynomials  $C|f^\infty$  similarly as in [7], at the expense of a term of size  $q^{g(1+\epsilon)}$ . Since

$$\sum_{C|f^\infty} \frac{1}{|C|^2 z^{d(C)}} = \prod_{P|f} \left( 1 - \frac{1}{|P|^2 z^{d(P)}} \right)^{-1},$$

we have

$$S_{2g}(V = \square) = q^{2g+1} \frac{1}{2\pi i} \oint_{|z|=r_1} \frac{(qz-1)z^g}{1-z} \sum_{l \in \mathcal{M}} z^{d(l)} \sum_{\substack{f \in \mathcal{M}_{\leq 2g} \\ d(f) \text{ even}}} \frac{1}{|f|z^{\frac{d(f)}{2}}} \frac{d_2(f)G(l^2, \chi_f)}{\sqrt{|f|} \prod_{P|f} \left( 1 - \frac{1}{|P|^2 z^{d(P)}} \right)} \left( 1 - \frac{1}{qz} \right) dz \\ + O(q^{g(1+\epsilon)}).$$

Using Lemma 4.2, equation (3.5) and Lemma 4.3, it follows that

$$S_{2g}(V = \square) = q^{2g+1} \left( \frac{1}{2\pi i} \right)^2 \oint_{|z|=r_1} \oint_{|w|=r_2} \frac{(qz-1)z^g(q^2w^2z)^{-g}}{(1-z)w(1-qw)^2(1-q^2w^2z)} \mathcal{R}(z, w) \left( 1 - \frac{1}{qz} \right) dw dz + O(q^{g(1+\epsilon)}),$$

where recall that  $r_1 = q^{-1-\epsilon}$  and  $r_2 < 1/q$ . Using Lemma 4.3 again, we have

$$S_{2g}(V = \square) = -q^{2g+1} \left( \frac{1}{2\pi i} \right)^2 \oint_{|z|=r_1} \oint_{|w|=r_2} \frac{z^g(q^2w^2z)^{-g} \mathcal{Z}(1/(q^2z)) \mathcal{F}(z, w)}{(1-z)w(1-qw)^2(1-q^2w^2z)^2} \left( 1 - \frac{1}{qz} \right) dw dz + O(q^{g(1+\epsilon)}).$$

From Lemma 4.3,  $\mathcal{Z}(1/(q^2z)) \mathcal{F}(z, w)$  is absolutely convergent for  $|w| < 1/\sqrt{q}$ ,  $|wz| < 1/q$  and  $|w^2z| < 1/q^2$ , so in the integral above we can shift the contour  $|z| = q^{-1-\epsilon}$  to  $|z| = q^{\epsilon-1}$  without encountering any poles. Then we have

$$S_{2g}(V = \square) = -q \left( \frac{1}{2\pi i} \right)^2 \oint_{|z|=r_1} \oint_{|w|=r_2} \frac{\mathcal{F}(z, w)}{(1-z)(1-qw)^2(1-q^2w^2z)^2 w^{2g+1}} dw dz + O(q^{g(1+\epsilon)}),$$

where  $r_1 = q^{\epsilon-1}$  and  $r_2 < 1/q$ . Enlarging the contour of integration  $|w| = r_2$  to  $|w| = q^{-1/2-\epsilon}$ , we encounter a double pole at  $w = 1/q$ , and the double integral  $\left( \frac{1}{2\pi i} \right)^2 \oint_{|z|=r_1} \oint_{|w|=q^{-1/2-\epsilon}}$  will be

bounded by  $q^{g(1+\epsilon)}$ . We compute the residue at  $w = 1/q$  and using the notation from Lemma 4.4 we have

$$S_{2g}(V = \square) = -q^{2g+1} \oint_{|z|=\frac{q^\epsilon}{q}} \frac{\mathcal{F}(z)}{(1-z)^3} \left( 2g+1 - \frac{4z}{1-z} - \alpha(z) \right) dz + O(q^{g(1+\epsilon)}).$$

Similarly

$$S_{2g-1}(V = \square) = -q^{2g+1} \oint_{|z|=\frac{q^\epsilon}{q}} \frac{z\mathcal{F}(z)}{(1-z)^3} \left( 2g-1 - \frac{4z}{1-z} - \alpha(z) \right) dz + O(q^{g(1+\epsilon)}).$$

Combining the above and using the fact that  $S_2(V = \square) = S_{2g}(V = \square) + S_{2g-1}(V = \square)$ , it follows that

$$S_2(V = \square) = -q^{2g+1} \frac{1}{2\pi i} \oint_{|z|=\frac{q^\epsilon}{q}} \frac{\mathcal{F}(z)(1+z)}{(1-z)^3} \left( 2g+1 - \frac{6z}{1-z^2} - \frac{2z^2}{1-z^2} - \alpha(z) \right) dz + O(q^{g(1+\epsilon)}).$$

From Lemma 4.4, note that  $\mathcal{F}(z)$  is analytic for  $1/q < |z| < q$ . We can compute the integral above exactly using the symmetry properties of  $\mathcal{F}$  and  $\alpha$ , by making the change of variables  $z = 1/u$ . Combining the functional equations for  $\mathcal{F}(z)$  and  $\alpha(z)$  as given in Lemma 4.4 and using the fact that

$$\frac{1}{2\pi i} \oint_{|z|=\frac{q^\epsilon}{q}} \frac{z(1+z)\mathcal{F}'(z)}{(1-z)^3} dz = -\frac{1}{2\pi i} \oint_{|z|=\frac{q^\epsilon}{q}} \frac{\mathcal{F}(z)(z^2+4z+1)}{(1-z)^4} dz,$$

it follows that

$$\begin{aligned} & \frac{1}{2\pi i} \oint_{|z|=\frac{q^\epsilon}{q}} \frac{\mathcal{F}(z)(1+z)}{(1-z)^3} \left( 2g+1 - \frac{6z}{1-z^2} - \frac{2z^2}{1-z^2} - \alpha(z) \right) dz \\ &= -\frac{1}{2\pi i} \oint_{|z|=\frac{q^\epsilon}{q}} \frac{\mathcal{F}(z)(1+z)}{(1-z)^3} \left( 2g+1 - \frac{6z}{1-z^2} - \frac{2z^2}{1-z^2} - \alpha(z) \right) dz. \end{aligned}$$

Note that in the annulus between  $|z| = q^{\epsilon-1}$  and  $|z| = q^{1-\epsilon}$  there is only one pole of the integrand at  $z = 1$ . Hence from the identity above, we can explicitly evaluate the integral as

$$\frac{1}{2\pi i} \oint_{|z|=\frac{q^\epsilon}{q}} \frac{\mathcal{F}(z)(1+z)}{(1-z)^3} \left( 2g+1 - \frac{6z}{1-z^2} - \frac{2z^2}{1-z^2} - \alpha(z) \right) dz = -\frac{\text{Res}(z=1)}{2}.$$

Then

$$S_2(V = \square) = \frac{q^{2g+1}}{2} \text{Res}(z=1) + O(q^{g(1+\epsilon)}).$$

Computing the residue at  $z = 1$  gives that

$$S_2(V = \square) = \frac{q^{2g+1}}{\zeta(2)} P_2(2g+1) + O(q^{g(1+\epsilon)}),$$

where  $P_2(2g+1) = 2\zeta(2)\text{Res}(z=1)$ , and we compute it explicitly as

(4.1)

$$\begin{aligned} P_2(x) &= -x \frac{\zeta(2)}{2} (\mathcal{F}'(1) + \mathcal{F}''(1)) \\ &\quad - \zeta(2) \left[ 2\mathcal{F}'(1) + 4\mathcal{F}''(1) + \mathcal{F}^{(3)}(1) + \frac{\alpha(1)(\mathcal{F}'(1) + \mathcal{F}''(1))}{2} + \frac{\alpha'(1)(\mathcal{F}(1) + \mathcal{F}'(1))}{2} + \frac{\alpha''(1)\mathcal{F}(1)}{2} \right]. \end{aligned}$$

This finishes the proof of Lemma 4.1.

5. EVALUATING THE ERROR FROM NON-SQUARE  $V$ 

In this section, we will bound  $S_{2g}(V \neq \square)$  and  $S_{2g-1}(V \neq \square)$ . Recall that  $S_{2g}(V \neq \square) = S_{2g,o} + S_{2g,e}(V \neq \square)$ , with  $S_{2g,o}$  given by (2.2) and  $S_{2g,e}(V \neq \square)$  the sum over non-square polynomials  $V$  in (2.1). We will prove the following.

**Lemma 5.1.** *Using the same notation as before, we have*

$$S_{2g}(V \neq \square) \ll q^{g(1+\epsilon)},$$

and

$$S_{2g-1}(V \neq \square) \ll q^{g(1+\epsilon)}.$$

*Proof.* In equation (2.2), write  $S_{2g,o}$  as a difference of two terms, and let  $S_{1,o}$  denote the first term and  $S_{2,o}$  the second. We will bound the term  $S_{1,o}$  and then bounding  $S_{2,o}, S_{2g,e}(V \neq \square)$  will follow similarly. We use the fact that

$$\sum_{\substack{C \in \mathcal{M}_i \\ C|f^\infty}} \frac{1}{|C|^2} = \frac{1}{2\pi i} \oint_{|u|=r_1} \frac{1}{q^{2i}u^{i+1} \prod_{P|f} (1 - u^{d(P)})},$$

where  $r_1 < 1$ . If we let  $d(f) = n$  and  $d(C) = i$ , then

(5.1)

$$S_{1,o} = q^{2g+1} \sqrt{q} \frac{1}{2\pi i} \oint_{|u|=r_1} \sum_{\substack{n=0 \\ n \text{ odd}}}^{2g} q^{-n} \sum_{i=0}^g \frac{1}{q^{2i}u^{i+1}} \sum_{V \in \mathcal{M}_{n-2g-2+2i}} \sum_{f \in \mathcal{M}_n} \frac{d_2(f)G(V, \chi_f)|f|^{-1/2}}{\prod_{P|f} (1 - u^{d(P)})} du.$$

We express the sum over  $f$  as a contour integral and use Lemma 4.2 to write

$$\sum_{f \in \mathcal{M}_n} \frac{d_2(f)G(V, \chi_f)|f|^{-1/2}}{\prod_{P|f} (1 - u^{d(P)})} = \frac{1}{2\pi i} \oint_{|w|=r_2} \frac{\mathcal{L}(w, \chi_V)^2 \prod_P \mathcal{M}_P(V; 1/(q^2u), w)}{w^{n+1}} dw.$$

From Lemma 4.2, note that  $\prod_P \mathcal{M}_P(V; 1/(q^2u), w)$  converges for  $|wu| < 1/q, |w| < 1/\sqrt{q}$  and  $|u| < 1$ . We pick  $r_1 = q^{-\epsilon}$  and  $r_2 = q^{-1/2-\epsilon}$ . Let  $k$  be the least integer such that  $r_1^k r_2 < 1/q$ . Then we can write

$$\prod_P \mathcal{M}_P(V; 1/(q^2u), w) = \mathcal{L}(wu, \chi_V)^2 \mathcal{L}(wu^2, \chi_V)^2 \cdots \mathcal{L}(wu^{k-1}, \chi_V)^2 \mathcal{B}(V; w, u),$$

where  $\mathcal{B}(V; w, u)$  is given by a converging Euler product. Hence we bound

$$(5.2) \quad \left| \sum_{f \in \mathcal{M}_n} \frac{d_2(f)G(V, \chi_f)|f|^{-1/2}}{\prod_{P|f} (1 - u^{d(P)})} \right| \ll q^{n/2(1+\epsilon)} \left| \mathcal{L}(w, \chi_V) \cdots \mathcal{L}(wu^{k-1}, \chi_V) \right|^2.$$

Note that the degree of  $V$  is odd, so  $V$  can't be a square. Using theorem 3.3 in [1] and the remarks in the proof of Lemma 7.1 in [7], it follows that

$$|\mathcal{L}(wu^j, \chi_V)| \ll e^{\frac{n-2g+2i}{2 \log q(n/2-g+i)} + 4\sqrt{q(n-2g+2i)}},$$

for  $j \in \{0, \dots, k-1\}$ . Using the bound above and combining equations (5.2) and (5.1), we get that

$$S_{1,o} \ll q^{g(1+\epsilon)} e^{\frac{2gk}{\log q(g)} + 8\sqrt{2gqk}} \ll q^{g(1+\epsilon)}.$$

Hence  $S_{2g}(V \neq \square) \ll q^{g(1+\epsilon)}$ .  $\square$

## 6. PROOF OF THEOREM 1.1

Now we put together the results from the previous sections. Combining Lemma 3.1, equation (3.7), Lemma 4.1, equation (4.1) and Lemma 5.1, it follows that

$$\sum_{D \in \mathcal{H}_{2g+1}} L\left(\frac{1}{2}, \chi_D\right)^2 = \frac{q^{2g+1}}{\zeta(2)} P(2g+1) + O(q^{g(1+\epsilon)}),$$

where  $P(x) = P_1(x) + P_2(x)$  is the following degree 3 polynomial

$$\begin{aligned} P(x) = & x^3 \frac{\mathcal{B}(1/q)(1-q^{-1})}{24} + x^2 \left[ \frac{\mathcal{B}(1/q)(1+q^{-1})}{4} - \frac{\mathcal{B}'(1/q)(1-q^{-1})}{4q} \right] \\ & + x \left[ \frac{11\mathcal{B}(1/q)(1-q^{-1})}{24} + \frac{3\mathcal{B}'(1/q)(1-q^{-1})}{2q} - \frac{2\mathcal{B}'(1/q)}{q} + \frac{\mathcal{B}''(1/q)(1-q^{-1})}{2q^2} - \frac{\zeta(2)}{2}(\mathcal{F}'(1) + \mathcal{F}''(1)) \right] \\ & + \frac{\mathcal{B}(1/q)(1+q^{-1})}{4} - \frac{\mathcal{B}'(1/q)(1-q^{-1})}{4q} + \frac{2\mathcal{B}'(1/q)}{q^2} + \frac{2\mathcal{B}''(1/q)}{q^3} - \frac{\mathcal{B}^{(3)}(1/q)(1-q^{-1})}{3q^3} \\ (6.1) \quad & - \zeta(2) \left[ 2\mathcal{F}'(1) + 4\mathcal{F}''(1) + \mathcal{F}^{(3)}(1) + \frac{\alpha'(1)(\mathcal{F}(1) + \mathcal{F}'(1))}{2} + \frac{\alpha(1)(\mathcal{F}'(1) + \mathcal{F}''(1))}{2} + \frac{\mathcal{F}(1)\alpha''(1)}{2} \right] \end{aligned}$$

*Remark 2.* We can check that the answer above matches the conjectured result in (1.5). For a polynomial  $Q$ , let  $[x^i]Q$  denote the coefficient of  $x^i$  in the polynomial  $Q$ . We will check that  $[x^i]P = [x^i]R$  for all  $i \in \{0, 1, 2, 3\}$ , with  $R$  given by (1.5).

(i) We have that  $[x^3]P = \frac{\mathcal{B}(1/q)(1-q^{-1})}{24}$  and  $[x^3]R = \frac{A(0,0)}{24}$ . Using Lemma 3.2,

$$\mathcal{B}(1/q) = \prod_P \left( 1 + \frac{1-3|P|}{|P|(|P|+1)^2} \right).$$

The identity then easily follows upon noticing that  $\mathcal{B}(1/q) \prod_P (1 - |P|^{-2}) = A(0, 0)$  and that  $\prod_P (1 - |P|^{-2}) = \zeta(2)^{-1} = 1 - q^{-1}$ .

(ii) Using equation (6.1), we have  $[x^2]P = \frac{\mathcal{B}(1/q)(1+q^{-1})}{4} - \frac{\mathcal{B}'(1/q)(1-q^{-1})}{4q}$ , while  $[x^2]R = \frac{A(0,0)}{4} + \frac{A_1(0,0)+A_2(0,0)}{8 \log q}$ . We compute  $\mathcal{B}'(1/q) = -q\mathcal{B}(1/q)b_1$ , where

$$b_1 = \sum_P \frac{d(P)(3|P|^2 - 2|P| - 1)}{(|P|+1)(|P|^3 + 2|P|^2 - 2|P| + 1)}.$$

From the definition of  $A(1/2; z_1, z_2)$  in (1.4), we also compute that

$$\frac{A_1(0,0)}{\log q} = \frac{A_2(0,0)}{\log q} = A(0,0) \left( b_1 + \sum_P \frac{2d(P)}{|P|^2 - 1} \right) = A(0,0) \left( b_1 + \frac{2}{q-1} \right),$$

where the last identity comes from the expression of the logarithmic derivative of  $\zeta(s)$ . Combining all of the above will give the desired identity.

(iii) We have that

$$[x]P = \frac{11\mathcal{B}(1/q)(1-q^{-1})}{24} + \frac{3\mathcal{B}'(1/q)(1-q^{-1})}{2q} - \frac{2\mathcal{B}'(1/q)}{q} + \frac{\mathcal{B}''(1/q)(1-q^{-1})}{2q^2} - \frac{\zeta(2)}{2}(\mathcal{F}'(1) + \mathcal{F}''(1))$$

and

$$[x]R = \frac{11}{24}A(0,0) + \frac{A_1(0,0) + A_2(0,0)}{\log q} + \frac{A_{12}(0,0)}{2(\log q)^2}.$$

From the definition of  $\mathcal{B}(u)$ , we get that  $\mathcal{B}''(1/q) = q^2\mathcal{B}(1/q)(b_1^2 + b_1 + b_2)$ , with  $b_1$  as before and

$$b_2 = - \sum_P \frac{d(P)^2|P|(3 - 5|P| - 2|P|^2 - 14|P|^3 - |P|^4 + 3|P|^5)}{(|P| + 1)^2(|P|^3 + 2|P|^2 - 2|P| + 1)^2}.$$

Also  $\mathcal{F}(1)\zeta(2) = A(0,0)$ ,  $\mathcal{F}'(1) = 0$  and  $\mathcal{F}''(1) = \mathcal{F}(1)b_3$ , where

$$b_3 = - \sum_P \frac{d(P)^2|P|(2 - 4|P| + 4|P|^2 + 2|P|^3)}{(|P|^3 + 2|P|^2 - 2|P| + 1)^2}.$$

We compute

$$\frac{A_{12}(0,0)}{(\log q)^2} = A(0,0) \left( \left( b_1 + \frac{2}{q-1} \right)^2 + b_2 - b_3 - \sum_P \frac{4d(P)^2|P|^2}{(|P|^2 - 1)^2} \right),$$

and using the fact that  $\sum_P \frac{4d(P)^2|P|^2}{(|P|^2 - 1)^2} = \frac{4}{(q-1)^2} + \frac{4}{q-1}$ , we have

$$\frac{A_{12}(0,0)}{(\log q)^2} = A(0,0) \left( \left( b_1 + \frac{2}{q-1} \right)^2 + b_2 - b_3 - \frac{4}{(q-1)^2} - \frac{4}{q-1} \right).$$

Doing the computations, we can check that  $[x]P = [x]R$ .

(iv) From equation (6.1), we have

$$\begin{aligned} [x^0]P &= \frac{\mathcal{B}(1/q)(1+q^{-1})}{4} - \frac{\mathcal{B}'(1/q)(1-q^{-1})}{4q} + \frac{2\mathcal{B}'(1/q)}{q^2} + \frac{2\mathcal{B}''(1/q)}{q^3} - \frac{\mathcal{B}^{(3)}(1/q)(1-q^{-1})}{3q^3} \\ &\quad - \zeta(2) \left[ 2\mathcal{F}'(1) + 4\mathcal{F}''(1) + \mathcal{F}^{(3)}(1) + \frac{\alpha'(1)(\mathcal{F}(1) + \mathcal{F}'(1))}{2} + \frac{\alpha(1)(\mathcal{F}'(1) + \mathcal{F}''(1))}{2} + \frac{\mathcal{F}(1)\alpha''(1)}{2} \right]. \end{aligned}$$

Also from (1.5),

$$\begin{aligned} [x^0]R &= \frac{A(0,0)}{4} + \frac{1}{24\log q}(A_1(0,0) + A_2(0,0)) + \frac{1}{(\log q)^2}A_{12}(0,0) \\ &\quad - \frac{1}{12(\log q)^3}(A_{222}(0,0) - 3A_{122}(0,0) - 3A_{112}(0,0) + A_{111}(0,0)). \end{aligned}$$

We compute

$$\mathcal{B}^{(3)}(1/q) = -q^3\mathcal{B}(1/q)(b_1^3 + 3b_1b_2 + b_4 + 3b_1^3 + 3b_2 + 2b_1),$$

where

$$b_4 = \sum_P \frac{d(P)^3|P|(-3 + 6|P| - 3|P|^2 + 91|P|^3 - 41|P|^4 - 29|P|^5 - 57|P|^6 - 55|P|^7 - 8|P|^8 + 3|P|^9)}{(|P| + 1)^3(|P|^3 + 2|P|^2 - 2|P| + 1)^3}.$$

Also  $\mathcal{F}^{(3)}(1) = -3\mathcal{F}(1)b_3$ ,  $\alpha(1) = 2 \left( b_1 + \frac{2}{q-1} \right)$ ,

$$\alpha'(1) = b_5 = \sum_P \frac{4d(P)^2|P|}{|P|^3 + 2|P|^2 - 2|P| + 1},$$

and

$$\alpha''(1) = -b_5 + b_6,$$



where

$$b_6 = \sum_P \frac{4d(P)^3 |P| (|P|^3 + |P|^2 - 1)}{(|P|^3 + 2|P|^2 - 2|P| + 1)^2}.$$

Using (1.4),

$$\frac{A_{111}(0, 0)}{(\log q)^3} = \frac{A_{222}(0, 0)}{(\log q)^3} = A(0, 0) \left( \left( b_1 + \frac{2}{q-1} \right)^3 + 3 \left( b_1 + \frac{2}{q-1} \right) b_7 + b_8 \right),$$

where

$$b_7 = - \sum_P \frac{d(P)^2 |P| (5 - 21|P| + 32|P|^2 - 16|P|^3 - 5|P|^4 + 9|P|^5)}{(|P| - 1)^2 (|P|^3 + 2|P|^2 - 2|P| + 1)^2},$$

and

$$b_8 = \sum_P \frac{d(P)^3 |P| (9 - 46|P| + 81|P|^2 - 35|P|^3 - 43|P|^4 + 29|P|^5 + 35|P|^6 - 29|P|^7 - 2|P|^8 + 17|P|^9)}{(|P|^4 + |P|^3 - 4|P|^2 + 3|P| - 1)^3}.$$

Similarly we compute

$$\begin{aligned} \frac{A_{122}(0, 0)}{(\log q)^3} &= \frac{A_{112}(0, 0)}{(\log q)^3} = A(0, 0) \left( \left( b_1 + \frac{2}{q-1} \right)^3 + 2 \left( b_1 + \frac{2}{q-1} \right) \left( b_2 - b_3 - \frac{4}{(q-1)^2} - \frac{4}{q-1} \right) \right. \\ &\quad \left. + \left( b_1 + \frac{2}{q-1} \right) b_7 + b_9 \right), \end{aligned}$$

where

$$\begin{aligned} \frac{b_9}{2} &= \frac{b_8}{6} - \frac{b_6}{2} + \frac{b_4}{3} + \sum_P \frac{8d(P)^3 |P|^2 (|P|^2 + 1)}{3(|P|^2 - 1)^3} \\ &= \frac{b_8}{6} - \frac{b_6}{2} + \frac{b_4}{3} + \frac{8q(q+1)}{3(q-1)^3}. \end{aligned}$$

Combining all of the above will give the desired identity of coefficients.

## 7. PROOF OF THEOREM 1.2

Here we will prove Theorem 1.2. Computing the third moment is similar to the computation of the second moment, so we will skip some of the details.

Recall from section 2.4 that  $S_{3g} = M_{3g} + S_{3g}(V = \square) + S_{3g}(V \neq \square) + S_{3g,0} + O(q^{3g/2(1+\epsilon)})$ , and a similar expression holds for  $M_{3g-1}$ .

**7.1. Main term.** Here we focus on the main term  $M_{3g}$ . Recall that

$$(7.1) \quad M_{3g} = q^{2g+1} \left( 1 - \frac{1}{q} \right) \sum_{\substack{f \in \mathcal{M}_{\leq 3g} \\ f = \square}} \frac{d_3(f)}{|f|^{\frac{3}{2}}} \phi(f) \sum_{\substack{C \in \mathcal{M}_{\leq g-1} \\ C|f^\infty}} \frac{1}{|C|^2}.$$

We have the following.

**Lemma 7.1.** *With the same notation as before,*

$$M_{3g} + M_{3g-1} = \frac{q^{2g+1}}{\zeta(2)} Q_1(2g+1) + O(q^{g(1+\epsilon)}),$$

where  $Q_1$  is a polynomial of degree 6.

*Proof.* Similarly as in section 3, we rewrite

$$(7.2) \quad M_{3g} = \frac{q^{2g+1}}{\zeta(2)} \sum_{l \in \mathcal{M}_{\leq [\frac{3g}{2}]}} \frac{d_3(l^2)}{|l| \prod_{P|l} \left(1 + \frac{1}{|P|}\right)} + O(q^{g\epsilon}).$$

Let

$$\mathcal{A}_3(u) = \sum_{l \in \mathcal{M}} u^{d(l)} \frac{d_3(l^2)}{\prod_{P|l} \left(1 + \frac{1}{|P|}\right)}.$$

Using Euler products, we get that

$$\begin{aligned} \mathcal{A}_3(u) &= \prod_P \left(1 + \frac{u^{d(P)}(6 - 3u^{d(P)} + u^{2d(P)})}{\left(1 + \frac{1}{|P|}\right)(1 - u^{d(P)})^3}\right) \\ &= \mathcal{Z}(u)^6 \mathcal{B}_3(u), \end{aligned}$$

where

$$(7.3) \quad \mathcal{B}_3(u) = \prod_P \left(1 - \frac{6u^{d(P)} - (15 - 6|P|)u^{2d(P)} + (20 - 8|P|)u^{3d(P)} - (15 - 3|P|)u^{4d(P)} + 6u^{5d(P)} - u^{6d(P)}}{|P| + 1}\right).$$

From the expression of  $\mathcal{B}_3(u)$  above, note that it converges absolutely for  $|u| < \frac{1}{\sqrt{q}}$ . We can further write

$$\mathcal{B}_3(u) = \frac{\mathcal{Z}(u^4)^6}{\mathcal{Z}(u^2)^6} \mathcal{C}(u),$$

where  $\mathcal{C}(u)$  converges absolutely for  $|u| < \frac{1}{q^{1/3}}$ . From the above we see that  $\mathcal{B}_3(u)$  has an analytic continuation for  $|u| < \frac{1}{q^{1/3}}$ .

Now using (3.5) in (7.2), we get that

$$(7.4) \quad M_{3g} = \frac{q^{2g+1}}{\zeta(2)} \frac{1}{2\pi i} \oint_{|u|=r_1} \frac{\mathcal{B}_3(u)}{(1-qu)^7 (qu)^{[3g/2]}} \frac{du}{u} + O(q^{g\epsilon}),$$

where  $r_1 < 1/q$ . Similarly

$$(7.5) \quad M_{3g-1} = \frac{q^{2g+1}}{\zeta(2)} \frac{1}{2\pi i} \oint_{|u|=r_1} \frac{\mathcal{B}_3(u)}{(1-qu)^7 (qu)^{[(3g-1)/2]}} \frac{du}{u} + O(q^{g\epsilon}).$$

Note that in the two integrals above, by shifting the contour of integration to a circle around the origin of radius  $R = q^{-1/3-\epsilon}$ , we encounter a pole at  $u = 1/q$ . Since  $\mathcal{B}_3(u)$  has an analytic continuation for  $|u| < q^{-1/3}$ , we see that

$$M_{3g} = -\text{Res}(u = 1/q) + O(q^{g(1+\epsilon)}),$$

and a similar formula holds for  $M_{3g-1}$ . By computing the residues at  $u = 1/q$  for  $M_{3g}$  and  $M_{3g-1}$ , Lemma 7.1 follows.  $\square$

**7.2. Secondary main term.** Here we will evaluate  $S_3(V = \square) = S_{3g}(V = \square) + S_{3g-1}(V = \square)$ , with  $S_{3g}(V = \square)$  given by (2.3). We'll prove the following.

**Lemma 7.2.** *With the same notation as before, we have*

$$S_3(V = \square) = \frac{q^{2g+1}}{\zeta(2)} Q_2(2g+1) + O(q^{3g/2(1+\epsilon)}),$$

where  $Q_2$  is a polynomial of degree 6.

Before proving the above, we will first state two additional lemmas. We'll omit the proofs.

**Lemma 7.3.** *Let  $V$  be a monic polynomial in  $\mathbb{F}_q[x]$ . For  $|z| > 1/q^2$ , let*

$$\mathcal{A}(V; z, w) = \sum_{f \in \mathcal{M}} w^{d(f)} \frac{d_3(f)G(V, \chi_f)}{\sqrt{|f|} \prod_{P|f} \left(1 - \frac{1}{|P|^{2z d(P)}}\right)}.$$

Then we have

(a)

$$\mathcal{A}(V; z, w) = \mathcal{L}(w, \chi_V)^3 \prod_P \mathcal{H}_P(V; z, w),$$

where

$$\mathcal{H}_P(V; z, w) = \begin{cases} 1 + \frac{3(\frac{V}{P})w^{d(P)}}{|P|^{2z d(P)} - 1} + 3w^{2d(P)} - \frac{9w^{2d(P)}}{1 - \frac{1}{|P|^{2z d(P)}}} - \left(\frac{V}{P}\right)w^{3d(P)} + \frac{9(\frac{V}{P})w^{3d(P)}}{1 - \frac{1}{|P|^{2z d(P)}}} - \frac{3w^{4d(P)}}{1 - \frac{1}{|P|^{2z d(P)}}} & \text{if } P \nmid V \\ 1 + \left(1 - \frac{1}{|P|^{2z d(P)}}\right)^{-1} \sum_{i=1}^{\infty} \frac{w^{id(P)} d_3(P^i) G(V, \chi_{P^i})}{|P|^{i/2} \left(1 - \frac{1}{|P|^{2z d(P)}}\right)} & \text{if } P|V \end{cases}$$

(b) If  $V = l^2$  and  $l \in \mathcal{M}$ , then

$$\mathcal{A}(l^2; z, w) = \mathcal{Z}(w)^3 \prod_P \mathcal{A}_P(l^2; z, w),$$

where

$$\mathcal{A}_P(l^2; z, w) = \begin{cases} 1 + \frac{3w^{d(P)}}{|P|^{2z d(P)} - 1} + 3w^{2d(P)} - \frac{9w^{2d(P)}}{1 - \frac{1}{|P|^{2z d(P)}}} - w^{3d(P)} + \frac{9w^{3d(P)}}{1 - \frac{1}{|P|^{2z d(P)}}} - \frac{3w^{4d(P)}}{1 - \frac{1}{|P|^{2z d(P)}}} & \text{if } P \nmid l \\ (1 - w^{d(P)})^3 \left(1 + \sum_{i=1}^{\infty} \frac{w^{id(P)} d_3(P^i) G(l^2, \chi_{P^i})}{|P|^{i/2} \left(1 - \frac{1}{|P|^{2z d(P)}}\right)}\right) & \text{if } P|l \end{cases}$$

We also have the following.

**Lemma 7.4.** *Keeping the notation from the previous lemma, let*

$$C(z, w) = \sum_{l \in \mathcal{M}} z^{d(l)} \prod_P \mathcal{A}_P(l^2; z, w).$$

(a) Then

$$\mathcal{C}(z, w) = \mathcal{Z}(z) \mathcal{Z}(qw^2z)^6 \mathcal{Z}\left(\frac{1}{q^2z}\right) \frac{\mathcal{Z}(w^2z^2)^3}{\mathcal{Z}(wz)^3} \mathcal{H}(z, w),$$

where  $\mathcal{H}(z, w) = \prod_P \mathcal{H}_P(z, w)$ , and

$$\begin{aligned} \mathcal{H}_P(z, w) = & (1 - w^d)^3 (1 - |P|(w^2z)^d)^3 (1 + (wz)^d)^3 \left(1 + 3w^d + 3|P|(zw^2)^d + \frac{3w^{2d}}{|P|} - 3(zw)^d - \frac{1}{|P|^{2z d}} \right. \\ & \left. - 6(zw^2)^d + |P|(zw^3)^d - 3(zw^4)^d - |P|(z^2w^3)^d + 3|P|(z^2w^4)^d + |P|(z^2w^6)^d - |P|^2(z^3w^6)^d\right). \end{aligned}$$

(Here,  $d$  stands for  $d(P)$ .) Moreover,  $\mathcal{H}(z, w)$  converges absolutely for  $|w| < q^{-1/2}$ ,  $|zw| < q^{-1/2}$ ,  $|zw^2| < q^{-3/2}$ ,  $|z| > q^{-1}$ .

(b) We have

$$\begin{aligned} \mathcal{H}_P\left(z, \frac{1}{q}\right) &= \left(1 - \frac{1}{|P|}\right)^3 \left(1 + \frac{3}{|P|} + \frac{3}{|P|^3} - \frac{1}{|P|^2 z^{d(P)}} - \frac{5z^{d(P)}}{|P|^2} - \frac{4z^{2d(P)}}{|P|^2} - \frac{6z^{2d(P)}}{|P|^3} - \frac{8z^{2d(P)}}{|P|^5} + \frac{14z^{3d(P)}}{|P|^4}\right. \\ &\quad + \frac{6z^{3d(P)}}{|P|^6} + \frac{6z^{4d(P)}}{|P|^4} + \frac{6z^{4d(P)}}{|P|^7} - \frac{12z^{5d(P)}}{|P|^6} - \frac{8z^{5d(P)}}{|P|^8} - \frac{4z^{6d(P)}}{|P|^6} + \frac{6z^{6d(P)}}{|P|^7} + \frac{2z^{7d(P)}}{|P|^8} + \frac{3z^{7d(P)}}{|P|^{10}} \\ &\quad \left. + \frac{z^{8d(P)}}{|P|^8} - \frac{3z^{8d(P)}}{|P|^9} - \frac{z^{8d(P)}}{|P|^{11}} + \frac{z^{9d(P)}}{|P|^{10}}\right), \end{aligned}$$

and  $\mathcal{H}\left(z, \frac{1}{q}\right)$  converges absolutely for  $q^{-1} < |z| < \sqrt{q}$  and has an analytic continuation when  $q^{-1} < |z| < q$ .

*Proof of Lemma 7.2.* Recall that

$$\begin{aligned} S_{3g}(V = \square) &= q^{2g+1} \sum_{\substack{f \in \mathcal{M}_{\leq 3g} \\ d(f) \text{ even}}} \frac{d_3(f)}{|f|^{\frac{3}{2}}} \sum_{\substack{C \in \mathcal{M}_{\leq g-1} \\ C|f^\infty}} |C|^{-2} \left[ (q-1) \sum_{l \in \mathcal{M}_{\leq \frac{d(f)}{2} - g - 2 + d(C)}} G(l^2, \chi_f) - \sum_{l \in \mathcal{M}_{\frac{d(f)}{2} - g - 1 + d(C)}} G(l^2, \chi_f) \right. \\ &\quad \left. - \frac{q-1}{q} \sum_{l \in \mathcal{M}_{\leq \frac{d(f)}{2} - g - 1 + d(C)}} G(l^2, \chi_f) + \frac{1}{q} \sum_{l \in \mathcal{M}_{\frac{d(f)}{2} - g + d(C)}} G(l^2, \chi_f) \right]. \end{aligned}$$

We proceed similarly as in section 4, and after using Lemmas 7.3 and 7.4, we get that

$$S_{3g}(V = \square) = -q^{2g+1} \frac{1}{(2\pi i)^2} \oint_{|z|=r_1} \oint_{|w|=r_2} \frac{z^g (q^2 w^2 z)^{-[3g/2]} (1 - qwz)^3}{w(1-z)(1-qw)^3 (1 - q^2 w^2 z)^7 (1 - qw^2 z^2)^3} \mathcal{H}(z, w) dw dz + O(q^{3g/2(1+\epsilon)}),$$

where  $r_2 < 1/q$  and  $r_1 = q^{\epsilon-1}$ . We enlarge the contour  $|w| = r_2$  to  $|w| = q^{-1/2-\epsilon}$ , and we encounter a pole of order 3 at  $w = 1/q$ . By Lemma 7.4,  $\mathcal{H}(z, w)$  is analytic in this region. When  $r_1 = q^{\epsilon-1}$  and  $|w| = q^{-1/2-\epsilon}$ , the double integral is bounded by  $O(q^{-g(1-\epsilon)})$ .

We evaluate the residue at  $w = 1/q$ , and we get that

$$S_{3g}(V = \square) = -q^{2g+1} \frac{1}{2\pi i} \oint_{|z|=q^{\epsilon-1}} \frac{\mathcal{H}(z, 1/q)}{z^{[3g/2]-g}(z-1)^7} P_1(z, g) dz,$$

where  $P_1(z, g)$  is a polynomial in  $z$  and  $g$ . In the expression for  $S_{3g}(V = \square)$  above, the integrand has a pole of order 7 at  $z = 1$ . By Lemma 7.4,  $\mathcal{H}(z, 1/q)$  is analytic for  $q^{-1} < |z| < \sqrt{q}$  and has an analytic continuation when  $|z| < q$ . By shifting the contour of integration to  $|z| = q^{1-\epsilon}$ , we encounter the pole at  $z = 1$  and we bound the integral over the new contour by  $O(q^{3g/2(1+\epsilon)})$ . We do the same for  $S_{3g-1}(V = \square)$ , and adding the two terms gives that  $S_3(V = \square) = \frac{q^{2g+1}}{\zeta(2)} Q_2(2g+1) + O(q^{3g/2(1+\epsilon)})$ , where the polynomial  $Q_2$  has degree 6 and can be computed explicitly by evaluating the residue at  $z = 1$ . This finishes the proof of Lemma 7.2.  $\square$

**7.3. Error from non-square  $V$ .** Here we'll show that  $S_{3g}(V \neq \square) \ll q^{3g/2(1+\epsilon)}$ . The proof is similar to the one in section 5. It is enough to bound the term  $S_{3g,o}$  given by (2.2) (bounding  $S_{3g,e}(V \neq \square)$  follows in the same way.) In equation (2.2), we write  $S_{3g,o}$  as a difference of two terms. Similarly as in section 5, we want to bound

$$(7.6) \quad S_{3,o} = q^{2g+1} \sqrt{q} \frac{1}{2\pi i} \oint_{|u|=r_1} \sum_{\substack{n=0 \\ n \text{ odd}}}^{3g} q^{-n} \sum_{i=0}^g \frac{1}{q^{2i} u^{i+1}} \sum_{V \in \mathcal{M}_{n-2g-2+2i}} \sum_{f \in \mathcal{M}_n} \frac{d_3(f) G(V, \chi_f) |f|^{-1/2}}{\prod_{P|f} (1 - u^{d(P)})} du,$$

where  $r_1 < 1$ . By Lemma 7.3,

$$\sum_{f \in \mathcal{M}_n} \frac{d_3(f)G(V, \chi_f)|f|^{-1/2}}{\prod_{P|f} (1 - u^{d(P)})} = \frac{1}{2\pi i} \oint_{|w|=r_2} \frac{\mathcal{L}(w, \chi_V)^3 \prod_P \mathcal{H}_P(V; 1/(q^2 u), w)}{w^{n+1}} dw,$$

where  $\prod_P \mathcal{H}_P(V; 1/(q^2 u), w)$  converges for  $|wu| < 1/q$ ,  $|w| < 1/\sqrt{q}$  and  $|u| < 1$ . We pick  $r_1 = q^{-\epsilon}$  and  $r_2 = q^{-1/2-\epsilon}$ , and let  $k$  be minimal such that  $r_1^k r_2 < 1/q$ . Then

$$\prod_P \mathcal{H}_P(V; 1/(q^2 u), w) = \mathcal{L}(wu, \chi_V)^3 \mathcal{L}(wu^2, \chi_V)^3 \cdots \mathcal{L}(wu^{k-1}, \chi_V)^3 \mathcal{D}(V; w, u),$$

where  $\mathcal{D}(V; w, u)$  is given by a converging Euler product. Then

$$\left| \sum_{f \in \mathcal{M}_n} \frac{d_3(f)G(V, \chi_f)|f|^{-1/2}}{\prod_{P|f} (1 - u^{d(P)})} \right| \ll q^{n/2(1+\epsilon)} |\mathcal{L}(w, \chi_V) \cdots \mathcal{L}(wu^{k-1}, \chi_V)|^3.$$

Combining this with the bound

$$|\mathcal{L}(wu^j, \chi_V)| \ll e^{\frac{n-2g+2i}{2 \log q(n/2-g+i)} + 4\sqrt{q(n-2g+2i)}},$$

for  $j \in \{0, \dots, k-1\}$ , and trivially bounding the sum over  $V$  gives that  $S_{3,o} \ll q^{3g/2(1+\epsilon)}$ . Then  $S_{3g}(V \neq \square) \ll q^{3g/2(1+\epsilon)}$ . Combining this bound with Lemmas 7.2 and 7.1 and putting  $Q(x) = Q_1(x) + Q_2(x)$  finishes the proof of Theorem 1.2.

*Remark 3.* We note that

$$[x^6]Q_1 = \frac{729}{2^{11}6!} \mathcal{B}_3(1/q),$$

which follows from evaluating the residues in the integrals (7.4) and (7.5). Also

$$[x^6]Q_2 = -\frac{217}{2^{11}6!} \mathcal{H}(1, 1/q) \zeta(2)^4,$$

which follows from evaluating the residue at  $z = 1$  in the integral for  $S_{3g}(V = \square)$  above. By direct computation, we have that  $\mathcal{B}_3(1/q) = \mathcal{H}(1, 1/q) \zeta(2)^4 = A_3(\frac{1}{2}; 0, 0, 0)$ , where  $A_3(\frac{1}{2}; 0, 0, 0)$  is given by equation (1.7). Combining the two equations above and since  $Q = Q_1 + Q_2$ , we have

$$[x^6]Q = \frac{1}{2880} A_3\left(\frac{1}{2}; 0, 0, 0\right),$$

which matches the leading coefficient in the conjectured formula (1.6).

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## REFERENCES

- [1] Salim Ali Altuğ and Jacob Tsimerman. Metaplectic Ramanujan conjecture over function fields with applications to quadratic forms. *Int. Math. Res. Not. IMRN*, (13):3465–3558, 2014.
- [2] J. C. Andrade and J. P. Keating. The mean value of  $L(\frac{1}{2}, \chi)$  in the hyperelliptic ensemble. *J. Number Theory*, 132(12):2793–2816, 2012.
- [3] J. C. Andrade and J. P. Keating. Conjectures for the integral moments and ratios of  $L$ -functions over function fields. *J. Number Theory*, 142:102–148, 2014.
- [4] Julio C. Andrade and Jonathan P. Keating. Mean value theorems for  $L$ -functions over prime polynomials for the rational function field. *Acta Arith.*, 161(4):371–385, 2013.
- [5] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith. Integral moments of  $L$ -functions. *Proc. London Math. Soc. (3)*, 91(1):33–104, 2005.
- [6] Adrian Diaconu, Dorian Goldfeld, and Jeffrey Hoffstein. Multiple Dirichlet series and moments of zeta and  $L$ -functions. *Compositio Math.*, 139(3):297–360, 2003.
- [7] Alexandra Florea. Improving the error term in the mean value of  $L(1/2, \chi_D)$  in the hyperelliptic ensemble. *preprint*.
- [8] G. H. Hardy and J. E. Littlewood. Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes. *Acta Math.*, 41(1):119–196, 1916.
- [9] D. R. Hayes. The expression of a polynomial as a sum of three irreducibles. *Acta Arith.*, 11:461–488, 1966.
- [10] A.E. Ingham. Mean-value theorems in the theory of the Riemann zeta-function. *Proc. London Math. Soc.*, 27:273–300, 1926.
- [11] M. Jutila. On the mean value of  $L(\frac{1}{2}, \chi)$  for real characters. *Analysis*, 1(2):149–161, 1981.
- [12] J. P. Keating and N. C. Snaith. Random matrix theory and  $L$ -functions at  $s = 1/2$ . *Comm. Math. Phys.*, 214(1):91–110, 2000.
- [13] J. P. Keating and N. C. Snaith. Random matrix theory and  $\zeta(1/2 + it)$ . *Comm. Math. Phys.*, 214(1):57–89, 2000.
- [14] Michael Rosen. *Number theory in function fields*, volume 210 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.
- [15] M. O. Rubinstein and K. Wu. Moments of zeta functions associated to hyperelliptic curves over finite fields. *preprint*, July 2014.
- [16] K. Soundararajan. Nonvanishing of quadratic Dirichlet  $L$ -functions at  $s = \frac{1}{2}$ . *Ann. of Math. (2)*, 152(2):447–488, 2000.
- [17] André Weil. *Sur les courbes algébriques et les variétés qui s'en déduisent*. Actualités Sci. Ind., no. 1041, Publ. Inst. Math. Univ. Strasbourg 7. Hermann et Cie., Paris, 1948.
- [18] Matthew P. Young. The third moment of quadratic Dirichlet  $L$ -functions. *Selecta Math. (N.S.)*, 19(2):509–543, 2013.
- [19] Qiao Zhang. On the cubic moment of quadratic Dirichlet  $L$ -functions. *Math. Res. Lett.*, 12(2-3):413–424, 2005.

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